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FORMULATION AND SOLUTION OF THE PROBLEM OF DETERMINING EROSION AND DEPOSITION IN OPEN CHANNELS FOR LOOSE SOILS†

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Flows of water and of the solid particles suspended in them are considered in a one-dimensional approximation and a formulation and solution of the problem of erosion and deposition in open channels are given for the case of loose soils. The effects of friction in the hydrodynamic equations are represented using the classical approach without invoking the concepts of a hydraulic radius and a Chézy coefficient. The rate of erosion and deposition are related to the flow parameters, and the effect of erosion and deposition on the hydrodynamic flow parameters is demonstrated. A theoretical description of this phenomenon, which is due to the removal of solid particles from the surface of the bottom into the flow and their deposition from the flow onto the bottom, is given. An example of a numerical calculation of the erosion and deposition in the case of the corresponding flow parameters is presented. The relations obtained enable the phenomena of erosion and deposition to be analysed both in the one-dimensional and the two-dimensional approximations.

The problem of erosion and deposition in the channels of rivers and reservoirs is of great practical importance, for example, when analysing erosion and deposition in channels and reservoirs during the operation of hydrotechnical installations, which can be the cause of the partial or complete exposure of such installations. There are also ecological aspects of this phenomenon associated with rivers becoming shallower by the removal of solid particles suspended in the flow (solid particle drain).

The solution of this problem consists of determining the hydrodynamic characteristics of the flow, on the basis of which the rate of removal and deposition (mass transfer) is estimated. This, in turn, leads to a change in the hydrodynamic characteristics of the flow.

Sections of channels are considered which have a slight mean inclination of the bottom, that does not usually exceed values of 5×10^{-4}. The change in the absolute vertical coordinate of the free surface of the flow is commensurate with the depth of the channel. The modulus of the vertical component of the velocity vector is considerably less than the modulus of the velocity vector. The "shallow water" theory approximation is used to describe the hydrodynamics of the flow. In the equation of motion, it is necessary to give special attention to describing the term associated with the effect of friction. An expression associated with the hydraulic radius is often used, which is only defined in the one-dimensional case and only in the case of simple forms of the cross-section of channel [1]. Better results are obtained using the Chézy coefficient [2]. However, in practice, it is determined by natural measurements of the hydrodynamic quantities. The means that an investigator is compelled to solve the problem by determining the hydrodynamic characteristics experimentally. A second drawback is the attachment of the value of the Chézy coefficient to a certain narrow range of values of the channel flow velocity.

Sometimes, the effect of friction is described by a term which takes into account the inclination of the free surface of the flow [3, 4]. This approach is valid for a constant value of the flow velocity in time, a cross-section area which is constant along the longitudinal coordinate and a number of other conditions.

Empirical relations or recommendations derived on this basis [5] are used to determine the rate at which solid particles are carried out into the flow. No theoretical description of this process has been found in the scientific literature. Note that, for the steady-state case, there is an equation for the mass

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balance of the solid particles moving in the flow disregarding their dimensions [6]. Equations will be written for the one-dimensional case which describe the hydrodynamics of the flow in an open channel. The term describing the effect of friction on the flow is determined from experiments based on classical hydrodynamics. It is shown that the friction depends on the erosion and deposition on the bottom surface. The fundamentals of a theory of erosion and deposition in open channels are proposed.

1. OBJECT OF THE INVESTIGATION. BASIC ASSUMPTIONS

A mixture of water and suspended solid particles in a section of an open channel length L is considered. We introduce a Cartesian system of coordinates. The z axis is directed upwards, the x axis is directed along the flow and the y axis completes the system of coordinates. The depth of the flow $h = z_+ - z_-$, where $z_- = z_-(x, y)$ is the absolute vertical coordinate of the bottom surface and $z_+ = z_+(x, y)$ is the absolute vertical coordinate of the flow free surface.

The flow mean velocity, which is equal to the ratio of the flow rate Q to the cross-section are S, as a rule has a value of the order of 1 m s^{-1}. The steady rate of deposition of particles in the quiescent water (the hydraulic size of the particles is w) does not usually exceed 0.01 m s^{-1} and can serve for estimating the rate of diffusion. In channel flows, the relative volume concentration of particle does not usually exceed 2%. Taking this into account, we will adopt the diffusion approximation when treating the motion of a two-phase medium in which the velocities of the two-phase flow and of the particles are identical.

It follows from estimates which have been made that the flow density can be assumed to be constant and equal to the water density ρ. We assume that the solid particles do not interact with one another either in the flow of the mixture or on the channel bottom (the approximation of a loose bottom soil). A diameter distribution function is specified for particles which are below the bottom surface. Furthermore, we will assume that the magnitude of the shear stress on the flow surface is much less than on the bottom.

Usually, in the case of multiphase media, the equations for each phase are written out separately and the equations for the mixture as a whole are obtained by summing over the phases. Here, the equations will be written out for the mixture as a whole and separately for the solid particles.

It is assumed that the effects of friction are solely associated with the roughness of the bottom surface and the purely turbulent flow conditions. The process in which particles are carried out into the flow from the bottom and deposited onto the bottom are treated without taking into account "bottom particle" flows in which displacement of the particles along the surface occurs without the particles being carried out into the flow.

2. THE GENERAL FORM OF THE FUNDAMENTAL EQUATIONS OF HYDRODYNAMICS

The equation of continuity for a mixture when there are no mass sources and sinks has the form

$$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{2.1}$$

The equation of motion of the mixture in the diffusion approximation will also hold for each phase. It is obtained by integrating the one-dimensional equation of "shallow water" theory not over the depth, as is usually done, but over the whole cross-section area of the channel S. This is due to the fact that the cross-section does not always have a simple form (rectangular, etc.). The first two terms describing the acceleration take the form indicated earlier in [7]. The form of the terms describing the effect of pressure and the mean inclination of the bottom over the cross-section will differ from the generally accepted form. We then represent the equation of motion of the mixture as follows:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\frac{Q^2}{S} + g\frac{\partial}{\partial x}\oiint_S (z_+ - z_-)ds = -g\oiint_S \frac{\partial z_-}{\partial x}ds - \frac{\operatorname{sign} Q}{\rho}\oint_l \tau dl; \quad dl^2 = dx^2 + dy^2 \tag{2.2}$$

Here, g is the acceleration due to gravity, τ is the magnitude of the shear stress and l is the wetted perimeter.

3. DETERMINATION OF THE MAGNITUDE OF THE SHEAR STRESS

In the classical one-dimensional formulation of the problem within the framework of "shallow water" theory, it is impossible to determine the last term on the right-hand side of Eq. (2.2). To identify this, we shall extend the limits of the one-dimensional approach. We will apply the natural conditions.

The mean value of the Reynolds number $\mathrm{Re} = Vh/\nu$ (V is the mean velocity of the flow with respect to the depth and ν is the coefficient of kinematic viscosity of water) has a magnitude of the order of 10^6. The expressions for the drag coefficient, taking account of the roughness of the bottom (of the mean statistical diameter of the particles $\bar{d}_b$ on the bottom), is taken for turbulent conditions in the form [8]

$$\lambda = 0.01375\alpha\left(\frac{\bar{d}_b}{2h} + \frac{34}{\mathrm{Re}}\right)^{0.25} \tag{3.1}$$

The coefficient α reflects the fact that there are laminar conditions in a small part of the flow (close to the channel boundaries). The modulus of the shear stress τ is related to the drag coefficient λ [9] as follows:

$$\frac{|\tau|}{\rho} = u_\tau^2 = \lambda\left(\mathrm{Re}, \frac{\bar{d}_b}{h}\right)V^2 \tag{3.2}$$

where u_τ is the modulus of the shear stress velocity.

In accordance with experimental data, we take the following dependence of the mean velocity over the depth on the depth:

$$V = \alpha_V h^n, \quad \alpha_V = Q/\oint\!\!\oint_S (z_+ - z_-)^n ds \tag{3.3}$$

The value of n can be determined by processing the natural data or from existing relations (usually, $0.2 \le n \le 0.35$).

Finally, substituting expression (3.2) and (3.3) into Eq. (2.2) we obtain the final form of the motion equation.

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\frac{Q^2}{S} + g\frac{\partial}{\partial x}\oint\!\!\oint_S (z_+ - z_-)ds = -g\oint\!\!\oint_S \frac{\partial z_-}{\partial x}ds - \operatorname{sign} Q\oint_l \lambda\left(\mathrm{Re}, \frac{\bar{d}_p}{h}\right)V^2 dl \tag{3.4}$$

The system of equations (2.1), (3.4) completely defines the dynamics of the flow in the channel. Note that, knowing the values of z_+, the cross-section area of the channel S can be found.

The initial and boundary conditions have the form

$$\begin{aligned} &z_+(t = 0) = \varphi_z(x), \quad Q(t = 0) = \varphi_Q(x); \\ &R_1(z_+, Q, t, x = 0) = 0, \quad R_2(z_+, Q, t, x = L) = 0 \end{aligned} \tag{3.5}$$

In practice, the absolute values of the level of the flow of the mixture can be determined most accurately at the ends of the section and the following can therefore be taken as the boundary conditions:

$$z_+(x = 0) = \gamma_1(t), \quad z_+(x = L) = \gamma_2(t) \tag{3.6}$$

The proposed approach, unlike the one-dimensional approach, enables one to find the value of the mean velocity V and to obtain more accurate results.

4. FUNDAMENTALS OF THE THEORY OF THE EROSION AND DEPOSITION OF LOOSE SOILS ON THE BOTTOM

We shall now consider the processes associated with the deposition and ejection of solid particles. Some particles are carried out into the flow from the surface of loose soil and, at the same time, particles can be deposited onto the bottom. When the rate of deposition is greater than the rate of ejection, filling in occurs. Otherwise, one speaks of erosion.

On the basis of natural investigations of the diameter d of all the particles in loose bottom soils and in flow participating in mass transfer, we subdivide the particles into three groups. The finest particles, which are distributed throughout the whole depth of the flow and are all carried out from the bottom surface into the flow, belong to the first group. Natural data determine the maximum diameter d_1 of these particles from the condition [10]

$$w_1 = w(d_1) = k_1 u_\tau, \quad 0.3 \le k_1 \le 0.4 \tag{4.1}$$

The second group consists of coarser particles with a diameter greater than d_1 but less then d_2. The diameter d_2 is determined from the condition

$$w_2 = w(d_2) = k_2 u_\tau, \quad 0.9 \le k_2 \le 1.1 \tag{4.2}$$

The particles of the second group differ from the particles of the first group solely in that they are not distributed throughout the whole of the depth but constitute a layer close to the bottom. The third group consists of particles with a diameter which is greater than d_2. These particles are not carried out from the bottom, and if they occur in the flow, they are instantaneously deposited onto the bottom.

Since the processes which have been described are intimately associated with the magnitude of the diameter of the particles d, it is necessary to use the probability theory and mathematical statistics to describe them. Mass transfer in the flow is described by the equation of continuity for the solid particles close to the bottom and is determined by the source terms.

The motion of the mixture is accompanied by the diffusion of solid particles. The particle distribution function in the flow $G(d)$ tends to adopt the form of an equilibrium distribution function $G_p(d)$. According to the central limit theorem in probability theory, this distribution will be close to a normal distribution. The balance equation for the solid phase can be written as

$$\frac{\partial(G\rho_p)}{\partial t} + V\frac{\partial(G\rho_p)}{\partial x} = -\frac{\rho_p(G - G_p)}{T(d)} + j_+ - j_- \tag{4.3}$$

Here ρ_p is the reduced density of the particles, j_+ is the rate at which particles are carried out from the bottom and j_- is the rate at which particles are deposited from the flow.

The establishment time of the process will be

$$T(d) = \frac{1}{G(d)}\int_0^d t_p \bar{g}(b)db; \; \bar{g}(b) = \frac{\partial G(b)}{\partial b}, \; t_p = \begin{cases} \dfrac{h}{w_2 - w}, & d \le d_1 \\ \dfrac{h_2(d)}{w_2 - w}, & d_1 \le d \le d_2 \end{cases}, \; h_2(d) = h\left[\frac{w_2 - w(d)}{w_2 - w_1}\right]^2$$

where $\bar{g}(b)$ is the particle probability distribution density in the flow, t_p is the establishment time of the process for a particle of diameter d and $h_2(d)$ is the depth, measured from the bottom surface onto which particles of the second group are raised.

We will assume that the particles are deposited from the flow onto the bottom instantaneously, which is permissible in view of the smallness of $h_2(d)$. Then

$$j_- = \rho_p \chi(d - d_2)[G - G(d_2)]\delta(t) \tag{4.4}$$

where $\chi(d)$ is the Heaviside function and $\delta(t)$ is the delta function.

We will now determine the rate j_+ at which solid particles are carried out from the bottom. Suppose $\bar{h}$ is the change in the vertical coordinate of the bottom surface due to the ejection of particles. The probability distribution density of particles on the bottom is equal to $f(d)$. If the thickness of the boundary layer is equal to $\delta(n)$, the rate of change of the vertical coordinate will be

$$\frac{\partial \bar{h}}{\partial t} = -\frac{m\delta_n}{(1-p)h}\int_0^{d_2}[w_2 - w(\xi)]f(\xi)d\xi \times$$
$$\times\left\{1 + (1-p)\int_0^{-\bar{h}}\chi(d^*_{\max} - d_2)[1 - F(d_2)]\frac{da}{\bar{d}_p} + (1-p)\int_0^A[1 - G(d_2)]\frac{da}{\bar{d}_p}\right\}^{-1}, \quad \bar{d}_p = \int_0^{d^b_{\max}} \xi f(\xi)d\xi \tag{4.5}$$

where p is the porosity of the soil on the bottom, $\bar{d}_p$ is the mean statistical diameter of the particles on the bottom, $d^b_{\max}$ is the maximum diameter of the particles on the bottom, A is the increment in the vertical coordinate of the bottom due to the deposition of particles of the mixture from the flow, and m is the intermittence coefficient, which defines the fraction of the time of the process when large scale turbulent vortices are observed which, in the general case, depends on the Reynolds number, Re.

According to natural observation, m takes a value close to 0.5. The particle self-riprap phenomenon has been taken into account in relation (4.5), the meaning of which lies in the fact that the fraction of coarse particles on the bottom can increase with time. We shall differentiate between two forms of particle self-riprap: an internal self-riprap which is due to the fact that not all the particles are carried out into the flow (coarse particles remain on the bottom where, as time passes, their fraction in the distribution function increases) and an external self-riprap which is associated with the fact that, for the most part, coarse particles are deposited from the flow onto the bottom, which also leads to an increase in the fraction on the bottom. In Eq. (4.5), the first integral in the braces describes the effect of an internal self-riprap on the transport of particles and the second integral describes the effect of an external self-riprap. After each stage of the transport of particles from the bottom at the depth $\bar{h}$, in relation (4.5), instead of the distribution function $F(d)$, it is necessary to take the particle distribution function directly on the bottom, which takes into account the possible phenomenon of a self-riprap

$$\Theta(d) = \frac{F(d)+I_1(d)+I_2(d)}{1+I_1(d)+I_2(d)}$$

$$I_1(d) = (1-p)\int_0^{-\bar{h}} \chi(d-d_2)[F(d)-F(d_2)]\frac{da}{\bar{d}_b},\quad I_2(d) = (1-p)\int_0^{A} \chi(d-d_2)[G(d)-G(d_2)]\frac{da}{\bar{d}_b} \tag{4.6}$$

where $I_i(d)$ are terms which correspond to an internal ($i = 1$) and external ($i = 2$) self-riprap and, in Eq. (4.5), it is therefore necessary to substitute the derivative $\partial\Theta/\partial d$ instead of $f(d)$.

When account is taken of expression (4.6) and the remarks which have been made, we obtain the following equation for determining the coordinate of the bottom surface from relation (4.5)

$$\frac{\partial z_-}{\partial t} = \frac{\partial \bar{h}}{\partial t} + A\delta(t),$$

$$A = \frac{\rho_p}{\bar{\rho}_p(1-p)[1-G(d_2)]}\int_{d_2}^{d_{\max}} \chi(b-d_2)db\int_0^{H_2} dz\{\exp[-\alpha_p(z-z_-)]\bar{g}_b\} \tag{4.7}$$

where $\bar{\rho}_p$ is the true density of the particles and $H_2 = h_2(d_2)$.

An expression for A is obtained on the basis of numerous experiments, from which it is known that the reduced density distribution of the particles in the flow of a mixture has an exponential form with

$$\rho_p \sim \exp\lfloor -\alpha_p(z-z_-)\rfloor$$

Furthermore, the rate of inflow of the mass of particles into the flow, due to them being carried out from the bottom, taking into account the particle diameter distribution, is given by

$$j_+ = -\frac{\bar{\rho}_p\alpha_p(1-p)[\Theta(d)-\Theta(d_2)]\partial h}{[1-\exp(-\alpha_p h)]\Theta(d_2)\ \partial t} \tag{4.8}$$

Hence, expressions (4.4) and (4.8) completely define the mass transfer (4.3). In practice, it is necessary to substitute the diameter $d_{\max}$ into Eq. (4.3) for the distribution function and an equation in the density of particles in the flow is obtained. The left-hand side of equality (4.3) is then differentiated, taking into account the equation obtained for the density of the particles, to obtain an equation for the distribution function separately.

For the solution of the mass transfer problem, the initial and boundary conditions will be as follows:

$$\rho_p(t=0) = R_0(x),\quad G(t=0) = G_0(d,x);\ \rho_p(x=0) = R_x(t),\quad G(x=0) = G_x(d,t) \tag{4.9}$$

We will now define the stages of the solution of the problem on erosion and deposition in open channels.

1. The system of equation (2.1), (3.4) is solved and the quantities Q, z_+, S, V, u_τ are determined.
2. The effect of the flow on the erosion and deposition is determined from relations (4.1) and (4.2).
3. The following are determined when solving the mass transfer equations (4.3) and Eqs (4.5) and (4.7):
   (a) the change in the particles density in the flow and their diameter distribution function;
   (b) the diameter distribution function of the particles on the bottom surface on account of their ejection and deposition;
   (c) the new coordinates z_- of the bottom surface on account of the erosion and deposition;
   (d) the overall transport of the solid particles by the flow.
4. The values of S due to the change in z_- (the effect of erosion and deposition on the cross-section area of the channel) are determined.
5. The new mean statistical diameter of the particles $\bar{d}_p$ on the bottom surface is determined.

As an example, we will now consider a section of a channel of length $L = 10000$ m in the form of a rectangle of width $B = 500$ m. At the initial instant of time $t = 0$

$$z_-(0, 0) = 0, \quad z_-(0, L) = -2 \text{ m}$$

The initial vertical coordinate of the bottom surface changes monotonically from 0 to –2 m along the whole length of the section. We will assume that the mean flow velocity is independent of the depth. The initial flow rate is equal to 2500 $m^3\ s^{-1}$. All points of the bottom have the same characteristics regarding the maximum diameter of the particles forming the bottom, which is equal to 2.17 mm. The true density of the particles is 2650 kg m^{-3}. The diameter distribution function of the particles divided by the maximum $F(d/d^b_{max})$ is shown in Fig. 1.

The changes in the value of z_+ with time t for different values of x/L are shown in Fig. 2, with a step size of 0.25. The initial value $z_+ = 5$ m corresponds to $x/L = 0$: $z_+ = 3$ m – $x/L = 1$. The remaining values of z_+ for other x/L are arranged between them.

The reduced density of the solid particles ρ_p when $x = 0$ is taken equal to zero. The calculated values of the flow rate Q for $x = 0$ are given in Fig. 3 since the difference in the value of Q was insignificant for different values of x.

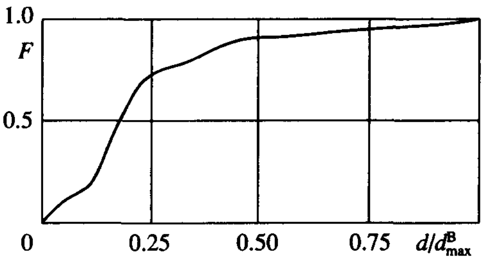

Fig. 1

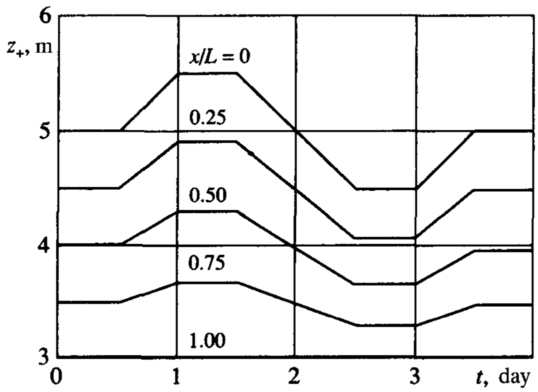

Fig. 2

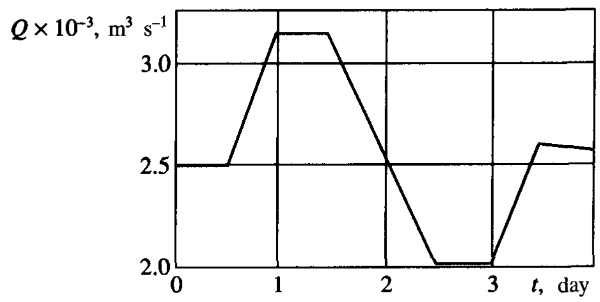

Fig. 3

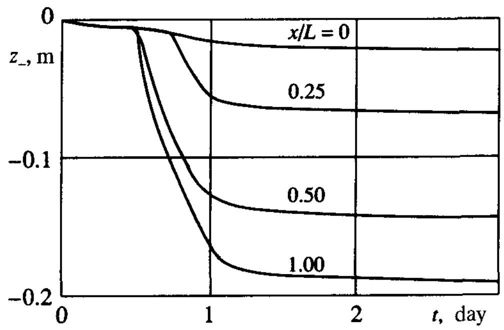

Fig. 4

The calculated values of the change in the vertical coordinate z_- of the points of the bottom surface are shown in Fig. 4. A reduction in the value of z_- corresponds to erosion and an increase to deposition. The lower curve corresponds to $x/L = 1$ and the upper curve to $x/L = 0$. The curves for other x/L are located between these two curves.

5. CONCLUSION AND RESULTS

1. A method for taking account of friction has been proposed which increases the accuracy with which the hydrodynamic characteristics of a flow can be determined.
2. Equations have been written, on the basis of the diffusion approximation for the one-dimensional problem, which describe the processes in open channels. Unlike the classical model, this model enable one to determine the longitudinal flow velocity.
3. Equations have been derived for the mass transfer between particles on the bottom surface and particles in the flow of the mixture. The rate at which particles are carried out from the bottom and the rate of deposition of particles from the flow onto the bottom have been determined as a function of the flow rate, the absolute coordinate of the water level and the absolute coordinate of the bottom surface.
4. The effect of the rate of the erosion and deposition of particles on the statistical characteristics of the particles and on the change in the coordinate in the bottom surface has been determined.
5. A theoretical description of the particle self-riprap phenomenon has been given.
6. A sequence of solutions of the problem has been presented.
7. An example of a numerical calculation of the parameters for erosion and deposition has been presented.

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